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The Poincaré compactification of the MIC–Kepler problem with positive energies

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Abstract

The Poincaré compactification and the symplectic reduction methods are first reviewed and then used to study the behaviour at infinity of the MIC (McIntosh–Cisneros)–Kepler problem at positive energies. The hyperbolic orbits leave the unstable equilibrium point set at infinity and tend eventually to the stable equilibrium point set at infinity. Both of these equilibrium point sets are diffeomorphic with S^2, the unit sphere in $\boldsymbol{R}^3$. The hyperbolic orbits determine a map of the unstable equilibrium point set to the stable equilibrium point set in such a manner that the initial point (or the limit point as $t \to -\infty$) of an orbit is mapped to its final point (or the limit point as $t \to \infty$). This map is found explicitly as a rotation matrix which depends on the energy and the angular momentum of the orbits.

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1. Introduction

Like the Kepler problem, the MIC (McIntosh–Cisneros)–Kepler problem, an extension of the Kepler problem [1], has orbits of conic sections in the configuration space $\boldsymbol{R}^3$. According to whether the energy is positive, zero, or negative, orbits are hyperbolae, parabolae, or ellipses. If one takes infinity into account, the hyperbola can be viewed as a curve leaving from one equilibrium point at infinity to another equilibrium point at infinity. This paper deals with the infinity of the MIC–Kepler problem through a Poincaré compactification [2–4] of polynomial vector fields. Here the Poincaré compactification is a topological method for forming the Poincaré sphere S^n by piecing together two copies of $\boldsymbol{R}^n$ at infinity through the central projection in such a manner that the equator of S^n corresponds to the infinity of $\boldsymbol{R}^n$. Polynomial vector fields on $\boldsymbol{R}^n$ can then be 'function-collinearly' extended to vector fields defined on the whole space S^n. This process is called the Poincaré compactification of polynomial vector fields. One cannot apply this compactification method to the MIC–Kepler problem in its original form, since the Hamiltonian vector field for the MIC–Kepler problem is not a polynomial vector field. However, the Poincaré compactification and the symplectic reduction method [5] applied to a certain polynomial Hamiltonian are put together to provide a method for treating the infinity of the MIC–Kepler problem with a positive energy. As a result, one can assume that hyperbolic orbits leave from unstable equilibrium points at infinity and eventually tend to stable equilibrium points at infinity. Thus hyperbolic orbits with a fixed energy value and a fixed angular momentum value determine a map of the stable equilibrium point set at infinity to the unstable equilibrium point set at infinity in such a manner that the initial point (or the limit point as $t \to -\infty$) of an orbit is mapped to the final point (or the limit point as $t \to \infty$). This map proves to be expressed as a rotation matrix:

$$\left(I + \frac{\sqrt{2h}}{\kappa} R(\boldsymbol{J})\right)^{-1} \left(I - \frac{\sqrt{2h}}{\kappa} R(\boldsymbol{J})\right) \tag{1.1}$$

where h is a positive energy value, κ is a positive constant, and the $R(\boldsymbol{J}) \in so(3)$ is the anti-symmetric matrix associated with the angular momentum $\boldsymbol{J}$.

This paper is organized as follows: sections 2 and 3 contain a review of the Poincaré compactification of polynomial vector fields and of the MIC–Kepler problem, respectively. The MIC–Kepler problem is formulated as a reduced Hamiltonian system of the conformal Kepler problem of four degrees of freedom [6, 7]. If the energy is fixed at a positive number, the conformal Kepler problem is replaced by a repulsive oscillator which has the same trajectories as the conformal Kepler problem within a change of parameters. In section 4, Poincaré compactification is applied to the repulsive oscillator to obtain a system on the sphere S^8. In section 5, trajectories of the conformal Kepler problem with a positive energy project to hyperbolic trajectories of the MIC–Kepler problem with a positive energy. The compactification procedure now allows one to deal with trajectories at infinity in an explicit manner. By the use of constants of motion for the MIC–Kepler problem along with the compactification procedure, one can find a map of the unstable equilibrium point set at infinity to the stable equilibrium point set at infinity, which is given by the rotation matrix (1.1).

2. A review of the Poincaré compactification

According to [3, 4], we make a brief review of the Poincaré compactification of polynomial vector fields. Let $X = (P_1, P_2, \ldots, P_n)$ be a polynomial vector field on $\boldsymbol{R}^n$ along with $m = \max\{\deg P_1, \ldots, \deg P_n\}$. We consider the Poincaré sphere $S^n = \{y \in \boldsymbol{R}^{n+1} \mid \sum_{i=1}^{n+1} y_i^2 = 1\}$ with the hyperplane $\Pi = \{y \in \boldsymbol{R}^{n+1} \mid y_{n+1} = 1\}$ tangent to S^n at the north pole. The Π is identified with $\boldsymbol{R}^n$ on which the polynomial vector field X is defined. Let H^+ and H^- be the open northern and southern hemispheres of S^n, respectively. Then the central projection of S^n to $\boldsymbol{R}^n$ defines the maps

$$\begin{aligned} &\psi^+ : \boldsymbol{R}^n \to H^+ \qquad \psi^+(x) = \frac{1}{\Delta(x)}(x_1, x_2, \ldots, x_n, 1) \\ &\psi^- : \boldsymbol{R}^n \to H^- \qquad \psi^-(x) = -\frac{1}{\Delta(x)}(x_1, x_2, \ldots, x_n, 1) \end{aligned} \qquad \Delta(x) = \left(1 + \sum_{j=1}^{n} x_j^2\right)^{1/2}. \tag{2.1}$$

The equator $S^{n-1} = \{y \in S^n | y_{n+1} = 0\}$ of the Poincaré sphere corresponds to the infinity of $\boldsymbol{R}^n$. Clearly, one has $S^n = H^+ \cup H^- \cup S^{n-1}$. The maps ψ^+ and ψ^- induce a vector field $\hat{X}$ on $H^+ \cup H^-$ through $\hat{X}(y) = (D\psi^+)_x X(x)$ for $y = \psi^+(x) \in H^+$ and through $\hat{X}(y) = (D\psi^-)_x X(x)$ for $y = \psi^-(x) \in H^-$, where $D\psi^+$ and $D\psi^-$ denote the differential map of ψ^+ and of ψ^-, respectively. A straightforward calculation provides

$$\hat{X}(y) = \left(y_{n+1} \sum_{j=1}^{n} (\delta_{ij} - y_i y_j) P_j, -y_{n+1}^2 \sum_{j=1}^{n} y_j P_j\right) \qquad \text{for} \quad y \in H^+ \cup H^- \tag{2.2}$$

where P_j is expressed as $P_j(\frac{y_1}{y_{n+1}}, \ldots, \frac{y_n}{y_{n+1}})$. A question now arises as to whether the $\hat{X}$ can be extended continuously to a vector field defined on the whole sphere S^n. For a possible extension, $\hat{X}(y)$ has to have a unique finite value as $y \in H^+ \cup H^-$ tends to each point of the equator. Unfortunately, there is no natural way to obtain such finite values for $\hat{X}(y)$ in general. However, if P_j are polynomials with $m = \max\{\deg P_1, \ldots, \deg P_n\}$, we can define an extendable vector field $\tilde{X}(y)$ by a function-collinear transformation of $\hat{X}(y)$:

$$\tilde{X}(y) = y_{n+1}^{m-1}\hat{X}(y) \qquad \text{for} \quad y \in H^+ \cup H^-. \tag{2.3}$$

In fact, since $\tilde{X}(y)$ takes the form

$$\tilde{X}(y) = \left(\sum_{j=1}^{n}(\delta_{ij} - y_i y_j)\tilde{P}_j, -y_{n+1}\sum_{j=1}^{n} y_j \tilde{P}_j\right) \tag{2.4}$$

with

$$\tilde{P}_j(y_1, \ldots, y_n, y_{n+1}) := y_{n+1}^m P_j\left(\frac{y_1}{y_{n+1}}, \ldots, \frac{y_n}{y_{n+1}}\right) \tag{2.5}$$

and since $\tilde{P}_j$ are all homogeneous polynomials of degree m in $y_1, \ldots, y_n, y_{n+1}$, each component of $\tilde{X}(y)$ has indeed a unique finite value as y tend to each point of the equator. Furthermore, equation (2.4) shows that $\tilde{X}(y)$ is tangent to the equator S^{n-1}, which implies that the equator is an invariant set under the flows generated by $\tilde{X}(y)$. Both H^+ and H^- are, of course, invariant sets. Thus we have observed that $\tilde{X}$ is defined on the whole sphere S^n in a natural manner. We also note that the vector fields $\hat{X}$ and $\tilde{X}$ share the same flows in $H^+ \cup H^-$ within a change of parameters; that is, flows defined by $\frac{dy}{d\tau} = \hat{X}(y)$ and by $\frac{dy}{dt} = \tilde{X}(y)$ are related by the change of parameters $\frac{d\tau}{dt} = y_{n+1}^{m-1}$. The vector field $\tilde{X}$ on S^n is called the Poincaré compactification of $X = (P_1, \ldots, P_n)$ on $\boldsymbol{R}^n$ (see also [2]). It is worth noting here that $\tilde{X}$ restricted on $H^+ \cup H^-$ is pulled back by ψ^+ and ψ^- to $[1 + \sum_{j=1}^n x_j^2]^{-\frac{m-1}{2}} X(x)$, the norm of which is of order $O(|x|)$ as $|x| \to \infty$ because of $m = \max\{\deg P_1, \ldots, \deg P_n\}$. Thus we obtain the 'compactified' differential equations on S^n:

$$\begin{aligned} \frac{dy_i}{dt} &= \sum_{j=1}^{n}(\delta_{ij} - y_i y_j)\tilde{P}_j(y_1, \ldots, y_n, y_{n+1}) \qquad i = 1, \ldots, n \\ \frac{dy_{n+1}}{dt} &= -y_{n+1}\sum_{j=1}^{n} y_j \tilde{P}_j(y_1, \ldots, y_n, y_{n+1}). \end{aligned} \tag{2.6}$$

In the case of the polynomial Hamiltonian vector field X_H associated with a polynomial H of degree $m + 1$ in $x_1, \ldots, x_d, x_{d+1}, \ldots, x_{2d}$ with $n = 2d$, one has

$$P_k = \frac{\partial H}{\partial x_{d+k}} \qquad P_{d+k} = -\frac{\partial H}{\partial x_k} \qquad k = 1, \ldots, d. \tag{2.7}$$

Then the corresponding polynomials (2.5) are shown to take the form

$$\tilde{P}_k = \frac{\partial \tilde{H}}{\partial y_{d+k}} \qquad \tilde{P}_{d+k} = -\frac{\partial \tilde{H}}{\partial y_k} \qquad k = 1, \ldots, d \tag{2.8}$$

along with

$$\tilde{H}(y_1, \ldots, y_n, y_{n+1}) = y_{n+1}^{m+1} H\left(\frac{y_1}{y_{n+1}}, \ldots, \frac{y_n}{y_{n+1}}\right). \tag{2.9}$$

Note here that $\tilde{H}$ is a homogeneous polynomial of degree $m + 1$ in $y_1, \ldots, y_n, y_{n+1}$. Hence equation (2.4) with $\tilde{P}^k$ and $\tilde{P}^{d+k}$ given by (2.7) provides the Poincaré compactification $\tilde{X}_H$

of X_H. It should be noted here that $\tilde{X}_H$ does not need to be the Hamiltonian vector field associated with $\tilde{H}$, while $\hat{X}_H$ may be viewed as a Hamiltonian vector field, if the standard symplectic form on $\boldsymbol{R}^{2d}$ is pushed forward to $H^+ \cup H^-$ by ψ^+ and ψ^-. Along with (2.8) and (2.9), equation (2.6) is brought into the form

$$\begin{aligned} \frac{\mathrm{d}y_k}{\mathrm{d}t} &= \frac{\partial \tilde{H}}{\partial y_{d+k}} + \lambda y_k \\ \frac{\mathrm{d}y_{d+k}}{\mathrm{d}t} &= -\frac{\partial \tilde{H}}{\partial y_k} + \lambda y_{d+k} \qquad k = 1, \ldots, d \\ \frac{\mathrm{d}y_{2d+1}}{\mathrm{d}t} &= \lambda y_{2d+1} \end{aligned} \tag{2.10}$$

with

$$\lambda = \sum_{j=1}^{d} \left(y_{d+j} \frac{\partial \tilde{H}}{\partial y_j} - y_j \frac{\partial \tilde{H}}{\partial y_{d+j}} \right). \tag{2.11}$$

Equation (2.10) shows that the infinity, the equator S^{2d-1} determined by $y \in S^{2d}$ and $y_{2d+1} = 0$, is an invariant set of the flow of $\tilde{X}_H$. To consider the flow at infinity, we express the polynomial Hamiltonian H as $H = H_0 + H_1 + \cdots + H_{m+1}$, where H_ℓ is the homogeneous part of H of degree ℓ for $\ell = 0, 1, \ldots, m + 1$. Then one has $\tilde{H} = H_{m+1}$ at infinity, so that equation (2.10) restricted at infinity becomes

$$\begin{aligned} \frac{\mathrm{d}y_k}{\mathrm{d}t} &= \frac{\partial H_{m+1}}{\partial y_{d+k}} + \lambda y_k \\ \frac{\mathrm{d}y_{d+k}}{\mathrm{d}t} &= -\frac{\partial H_{m+1}}{\partial y_k} + \lambda y_{d+k} \end{aligned} \qquad k = 1, \ldots, d \tag{2.12}$$

with

$$\lambda = \sum_{j=1}^{d} \left(y_{d+j} \frac{\partial H_{m+1}}{\partial y_j} - y_j \frac{\partial H_{m+1}}{\partial y_{d+j}} \right). \tag{2.13}$$

From equation (2.10), we also observe that the level set $\tilde{H}^{-1}(0)$ is an invariant set of $\tilde{X}_H$. In fact, on account of (2.10) and the homogeneity of $\tilde{H}$, one obtains

$$\frac{\mathrm{d}\tilde{H}}{\mathrm{d}t} = (m + 1)\lambda \tilde{H} \tag{2.14}$$

which shows that $\tilde{H}^{-1}(0)$ is an invariant set.

Proposition 1. *A polynomial vector field X on $\boldsymbol{R}^n$ is Poincaré compactified so as to be a vector field on S^n. In particular, a polynomial Hamiltonian vector field on $\boldsymbol{R}^{2d}$ is Poincaré compactified to give rise to equations of motion (2.10) on S^{2d}, which are no longer Hamilton's equations of motion, though the Hamiltonian H is extended to a function $\tilde{H}$ on S^n. Furthermore, the level set $\tilde{H}^{-1}(0)$ is an invariant set of the flows determined by (2.10), so that one can asymptotically approach infinity through $\tilde{H}^{-1}(0)$, if $\tilde{H}^{-1}(0)$ has a non-empty intersection with the equator of S^{2d}.*

3. A review of the MIC–Kepler problem

In this section, we make a brief review of the MIC–Kepler problem [6, 7], which is defined through the reduction method [5]. Let $\dot{\boldsymbol{R}}^4 := \boldsymbol{R}^4 - \{0\}$ be the configuration space with the Cartesian coordinates ξ_j, $j = 1, \ldots, 4$, and $T^*\dot{\boldsymbol{R}}^4 \cong \dot{\boldsymbol{R}}^4 \times \boldsymbol{R}^4$ be the cotangent bundle of $\dot{\boldsymbol{R}}^4$

with the Cartesian coordinates (ξ_j, η_k). The standard symplectic form $\mathrm{d}\theta$ on $T^*\dot{\boldsymbol{R}}^4$ is given by

$$\mathrm{d}\theta = \sum_{j=1}^{4} \mathrm{d}\eta_j \wedge \mathrm{d}\xi_j. \tag{3.1}$$

A $U(1) \cong SO(2)$ action on $\dot{\boldsymbol{R}}^4$ is defined through

$$\begin{aligned} (\xi_1 + \mathrm{i}\xi_2, \xi_3 + \mathrm{i}\xi_4) &\to \mathrm{e}^{\mathrm{i}t/2}(\xi_1 + \mathrm{i}\xi_2, \xi_3 + \mathrm{i}\xi_4) \\ (\eta_1 + \mathrm{i}\eta_2, \eta_3 + \mathrm{i}\eta_4) &\to \mathrm{e}^{\mathrm{i}t/2}(\eta_1 + \mathrm{i}\eta_2, \eta_3 + \mathrm{i}\eta_4) \end{aligned} \tag{3.2}$$

where $\mathrm{i} = \sqrt{-1}$. This action is clearly symplectic. The associated momentum map is then given by

$$\Phi(\xi, \eta) = \tfrac{1}{2}(-\xi_2\eta_1 + \xi_1\eta_2 - \xi_4\eta_3 + \xi_3\eta_4). \tag{3.3}$$

According to the reduction procedure, we take a momentum manifold $\Phi^{-1}(\mu)$ for $\mu \in \boldsymbol{R}$ fixed. For $\mu \neq 0$, $\Phi^{-1}(\mu)$ is a manifold. The reduced manifold is then defined to be a factor space, $P_\mu := \Phi^{-1}(\mu)/U(1)$. We denote the projection map by π_μ:

$$\pi_\mu : \Phi^{-1}(\mu) \longrightarrow P_\mu := \Phi^{-1}(\mu)/U(1). \tag{3.4}$$

The space P_μ is diffeomorphic with $T^*\dot{\boldsymbol{R}}^3 \cong \dot{\boldsymbol{R}}^3 \times \boldsymbol{R}^3$, where $\dot{\boldsymbol{R}}^3 = \boldsymbol{R}^3 - \{0\}$. The projection π_μ is realized as

$$\begin{aligned} q_1 &= 2(\xi_3\xi_1 + \xi_4\xi_2) \\ q_2 &= 2(-\xi_4\xi_1 + \xi_3\xi_2) \\ q_3 &= \xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2 \\ p_1 &= \frac{1}{2r}(\xi_3\eta_1 + \xi_4\eta_2 + \xi_1\eta_3 + \xi_2\eta_4) \\ p_2 &= \frac{1}{2r}(-\xi_4\eta_1 + \xi_3\eta_2 + \xi_2\eta_3 - \xi_1\eta_4) \\ p_3 &= \frac{1}{2r}(\xi_1\eta_1 + \xi_2\eta_2 - \xi_3\eta_3 - \xi_4\eta_4) \end{aligned} \tag{3.5}$$

where

$$r = \sum_{j=1}^{4} \xi_j^2 = \sqrt{\sum_{k=1}^{3} q_k^2}. \tag{3.6}$$

Let $\iota_\mu : \Phi^{-1}(\mu) \to T^*\dot{\boldsymbol{R}}^4$ be the inclusion map. Then the reduced symplectic form ω_μ on P_μ is determined through $\iota_\mu^*\mathrm{d}\theta = \pi_\mu^*\omega_\mu$, and found to be expressed as

$$\omega_\mu = \sum_{k=1}^{3} \mathrm{d}p_k \wedge \mathrm{d}q_k - \frac{\mu}{r^3}(q_1\,\mathrm{d}q_2 \wedge \mathrm{d}q_3 + q_2\,\mathrm{d}q_3 \wedge \mathrm{d}q_1 + q_3\,\mathrm{d}q_1 \wedge \mathrm{d}q_2). \tag{3.7}$$

Thus $(T^*\dot{\boldsymbol{R}}^4, \mathrm{d}\theta)$ reduces to $(T^*\dot{\boldsymbol{R}}^3, \omega_\mu)$.

The Hamiltonian system $(T^*\dot{\boldsymbol{R}}^4, \mathrm{d}\theta, H)$ with H defined as

$$H = \frac{1}{8r}\sum_{j=1}^{4} \eta^2 - \frac{\kappa}{r} \tag{3.8}$$

is called the conformal Kepler problem, where κ is a positive constant. This system is reduced to the Hamiltonian system $(T^*\dot{\boldsymbol{R}}^3, \omega_\mu, H_\mu)$, called the MIC–Kepler problem, where H_μ is determined through $H \circ \iota_\mu = H_\mu \circ \pi_\mu$, and expressed as

$$H_\mu = \frac{1}{2}\sum_{k=1}^{3} p_k^2 - \frac{\kappa}{r} + \frac{\mu^2}{2r^2}. \tag{3.9}$$

The conformal Kepler problem has the following constants of motion:

$$
\begin{aligned}
F_1 &= \tfrac{1}{2}(\xi_1\eta_4 - \xi_4\eta_1 + \xi_3\eta_2 - \xi_2\eta_3) \\
F_2 &= \tfrac{1}{2}(\xi_1\eta_3 - \xi_3\eta_1 + \xi_2\eta_4 - \xi_4\eta_2) \\
F_3 &= \tfrac{1}{2}(\xi_1\eta_2 - \xi_2\eta_1 + \xi_4\eta_3 - \xi_3\eta_4) \\
G_1 &= \tfrac{1}{4}(\eta_1\eta_3 + \eta_2\eta_4) - 2(\xi_1\xi_3 + \xi_2\xi_4)H \\
G_2 &= \tfrac{1}{4}(\eta_2\eta_3 - \eta_1\eta_4) - 2(\xi_2\xi_3 - \xi_1\xi_4)H \\
G_3 &= \tfrac{1}{8}(\eta_1^2 + \eta_2^2 - \eta_3^2 - \eta_4^2) - (\xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2)H.
\end{aligned}
\tag{3.10}
$$

These are $SO(2)$ invariant, so that they project to functions on the reduced phase space $T^*\dot{\boldsymbol{R}}^3$. The reduced functions J_k and A_k determined through $J_k \circ \pi_\mu = F_k \circ \iota_\mu$, $A_k \circ \pi_\mu = G_k \circ \iota_\mu$, $k = 1, 2, 3$, turn out to be expressed as

$$
\boldsymbol{J} = \boldsymbol{q} \times \boldsymbol{p} + \frac{\mu}{r}\boldsymbol{q} \qquad \boldsymbol{A} = \boldsymbol{J} \times \boldsymbol{p} + \frac{\kappa}{r}\boldsymbol{q}. \tag{3.11}
$$

Let $\{\ ,\ \}$ and $\{\ ,\ \}_\mu$ denote the Poisson brackets determined on $T^*\dot{\boldsymbol{R}}^4$ by $\mathrm{d}\theta$ and on $T^*\dot{\boldsymbol{R}}^3$ by ω_μ, respectively. Let K_μ and L_μ be the reduced function of K and L, respectively: $K \circ \iota_\mu = K_\mu \circ \pi_\mu$, $L \circ \iota_\mu = L_\mu \circ \pi_\mu$. Then one can show that

$$
\{K, L\} \circ \iota_\mu = \{K_\mu, L_\mu\}_\mu \circ \pi_\mu. \tag{3.12}
$$

Since $\{F_k, H\} = \{G_k, H\} = 0$, formula (3.12) applied to F_k and G_k implies that J_k and A_k are constants of motion for the MIC–Kepler problem; $\{J_k, H_\mu\}_\mu = \{A_k, H_\mu\}_\mu = 0$, $k = 1, 2, 3$. In analogy to the Kepler problem, the constants of motion $\boldsymbol{J}$ and $\boldsymbol{A}$ are called the angular momentum and the Runge–Lenz vector, respectively. It is to be noted here that the $\boldsymbol{J}$ has an excess term, $\frac{\mu}{r}\boldsymbol{q}$, in comparison with the ordinary angular momentum. This is because the symplectic form ω_μ also has an excess term, which plays the role of a magnetic field. If $\mu = 0$, these excess terms vanish, and the MIC–Kepler problem becomes the ordinary Kepler problem.

As is well known, one has $|\boldsymbol{J}|^2 \geqslant \mu^2$ from (3.11). If $|\boldsymbol{J}|^2 > \mu^2$, the orbits of the MIC–Kepler problem in the configuration space $\dot{\boldsymbol{R}}^3$ are conic sections [1]; they are intersections of the cone determined by $\boldsymbol{J} \cdot \boldsymbol{q}/r = \mu$ and the plane perpendicular to the constant vector $-\mu\boldsymbol{A} + \kappa\boldsymbol{J}$. These conic sections are ellipses, parabolae, or hyperbolae, according to whether the energy is negative, zero, or positive. If $|\boldsymbol{J}|^2 = \mu^2$, one has $\boldsymbol{q} \times \boldsymbol{p} = 0$, so that $-\mu\boldsymbol{A} + \kappa\boldsymbol{J} = 0$. In this case, the orbits are in straight lines through the origin.

Proposition 2. *The conformal Kepler problem* $(T^*\dot{\boldsymbol{R}}^4, \mathrm{d}\theta, H)$ *is reduced to the MIC–Kepler problem* $(T^*\dot{\boldsymbol{R}}^3, \omega_\mu, H_\mu)$*, which admits constants of motion* $\boldsymbol{J}$ *and* $\boldsymbol{A}$ *given in (3.11), and has orbits of conic sections in the configuration space* $\dot{\boldsymbol{R}}^3$.

4. The Poincaré compactification of the conformal Kepler problem

The Hamiltonian H of the conformal Kepler problem is not a polynomial, so that one cannot apply the Poincaré compactification method to the associated Hamiltonian vector field X_H. However, if we fix an energy value, say h, of the conformal Kepler problem, and if we consider the Hamiltonian defined and expressed as

$$
K := 4r(H - h) = \tfrac{1}{2}\sum_{j=1}^{4}\eta_j^2 - 4h\sum_{j=1}^{4}\xi_j^2 - 4\kappa \tag{4.1}
$$

instead of H, we can apply the Poincaré compactification method to K. We note here that r stands for the squared radius (see (3.6)). We also note that the Hamiltonian vector fields, X_H

and X_K, associated with H and K, respectively, share trajectories up to parameters. In fact, we obtain $4rX_H = X_K$ on the energy manifold $K^{-1}(0) = H^{-1}(h)$. Since we are interested in trajectories coming from or going to infinity, we assume that h is a non-negative constant.

On setting

$$\xi_j = \frac{y_j}{y_9} \qquad \eta_j = \frac{y_{4+j}}{y_9} \qquad j = 1, \ldots, 4 \tag{4.2}$$

the extended function $\tilde{K}$ defined in the same manner as in (2.9) takes the form

$$\tilde{K} = \frac{1}{2}\sum_{j=1}^{4} y_{4+j}^2 - 4h\sum_{j=1}^{4} y_j^2 - 4\kappa y_9^2. \tag{4.3}$$

Then the equations of motion (2.10) with $\tilde{H}$ replaced by $\tilde{K}$ are expressed, on S^8, as

$$\begin{aligned} \frac{\mathrm{d}y_j}{\mathrm{d}t} &= y_{4+j} + \lambda y_j \\ \frac{\mathrm{d}y_{4+j}}{\mathrm{d}t} &= 8hy_j + \lambda y_{4+j} \qquad j = 1, \ldots, 4 \\ \frac{\mathrm{d}y_9}{\mathrm{d}t} &= \lambda y_9 \end{aligned} \tag{4.4}$$

where

$$\lambda = -(8h+1)\sum_{j=1}^{4} y_j y_{4+j}.$$

From definition (2.9), $K = 0$ and $\tilde{K} = 0$ defines the same level sets in S^8 but outside S^7, where S^7 is the equator of S^8 defined by $y_9 = 0$. Since $\tilde{K}^{-1}(0)$ is invariant under the flow of (4.4), all trajectories $y(t)$ in $K^{-1}(0) \subset \tilde{K}^{-1}(0)$ tend to infinity S^7, as $t \to \pm\infty$. Furthermore, since every trajectory of the Hamiltonian vector field X_K with $h > 0$,

$$\xi_j(t) = a_j \mathrm{e}^{\sqrt{8h}t} + b_j \mathrm{e}^{-\sqrt{8h}t} \qquad \eta_j(t) = \sqrt{8h}(a_j \mathrm{e}^{\sqrt{8h}t} - b_j \mathrm{e}^{-\sqrt{8h}t}) \tag{4.5}$$

has a fixed direction in $\boldsymbol{R}^8$ as $t \to \pm\infty$, the corresponding trajectory $y(t)$ in S^8 will go to an equilibrium point in the equator S^7.

Since S^7 is also invariant under the flow of (4.4), we can consider the intersection $\tilde{K}^{-1}(0) \cap S^7$ as an invariant set at infinity, which turns out to be expressed as

$$E_h^{\infty} := \left\{ y \in \boldsymbol{R}^9 \,\middle|\, \sum_{j=1}^{4} y_j^2 = \frac{1}{8h+1}, \sum_{j=1}^{4} y_{4+j}^2 = \frac{8h}{8h+1}, y_9 = 0 \right\}. \tag{4.6}$$

This shows that $E_h^{\infty} \cong S^3 \times S^3$, if $h > 0$. The equilibrium points to which the above-mentioned trajectories $y(t)$ tend are in E_h^{∞}. From (4.4), the set of equilibrium points for $\tilde{X}_K$ in E_h^{∞} for $h > 0$ is determined by

$$y_j = -\frac{\lambda}{8h} y_{4+j} \qquad y_{4+j} = -\lambda y_j \qquad j = 1, \ldots, 4. \tag{4.7}$$

This implies that $\lambda = \pm\sqrt{8h}$. Thus we obtain the set of equilibrium points at infinity, which consists of two connected components both diffeomorphic with S^3:

$$S_\varepsilon^3 = \left\{ y \in \boldsymbol{R}^9 \,\middle|\, \sum_{j=1}^{4} y_j^2 = \frac{1}{8h+1}, y_{4+j} = \varepsilon\sqrt{8h}y_j, j = 1, \ldots, 4, y_9 = 0 \right\} \qquad \varepsilon = \pm. \tag{4.8}$$

From (4.2) and (4.5), one has

$$\frac{y_{4+j}(t)}{y_j(t)} \longrightarrow \pm\sqrt{8h} \qquad \text{as} \quad t \to \pm\infty. \tag{4.9}$$

This implies that the trajectories $y(t)$ lying in S^8 but outside S^7 tend to or leave from S^3_ε, depending on whether $\varepsilon = +$ or $-$.

We can show further that S^3_+ and S^3_- are stable and unstable, respectively, in the sense of linearization of $\tilde{X}_K$. To see this, we take the tangent plane to S^8 at a point $(1, 0, \ldots, 0)$ of the equator S^7. The central projection of S^8 to this tangent plane defines a local coordinate system in the open set $U_1 = \{y \in S^8 | y_1 > 0\}$ or in the open set $V_1 = \{y \in S^8 | y_1 < 0\}$. Take the local coordinates $(z_1, \ldots, z_8)$ defined by $z_i = y_{1+i}/y_1$, $i = 1, \ldots, 8$. Then the equations of motion (4.4) are put in the form

$$\frac{\mathrm{d}z_k}{\mathrm{d}t} = z_{4+k} - z_k z_4 \qquad \frac{\mathrm{d}z_4}{\mathrm{d}t} = 8h - z_4^2$$
$$\frac{\mathrm{d}z_{4+k}}{\mathrm{d}t} = 8hz_k - z_{4+k}z_4 \qquad \frac{\mathrm{d}z_8}{\mathrm{d}t} = -z_8 z_4 \qquad k = 1, 2, 3. \tag{4.10}$$

A straightforward calculation shows that the Jacobian matrix of the vector field (4.10) evaluated at any point of S^3_ε has eigenvalues 0, $-2\varepsilon\sqrt{8h}$, $-\varepsilon\sqrt{8h}$ with $\varepsilon = \pm 1$. While one has the eigenvalue 0, the associated eigenspace proves to be the tangent space to S^3_ε at the point of evaluation. Thus we see that S^3_ε is stable or unstable, depending on whether $\varepsilon = +$ or $-$. We note here that the eigenspaces associated with $-\varepsilon\sqrt{8h}$ and with $-2\varepsilon\sqrt{8h}$ are transversal to and tangent to the equator, respectively.

Proposition 3. *The conformal Kepler problem of a positive energy is Poincaré compactified through the function-collinear change of Hamiltonian vector fields,* $4rX_H = X_K$, *on the energy manifold* $H^{-1}(h) = K^{-1}(0)$. *The Poincaré compactified vector field* $\tilde{X}_K$ *has a stable and an unstable equilibrium point set* S^3_ε, *given in (4.8), at infinity.*

5. The Poincaré compactification of the MIC–Kepler problem

Thus far we have Poincaré compactified the conformal Kepler problem through the function-collinear change of the Hamiltonian vector fields, $4rX_H = X_K$, and thereby found the stable and the unstable equilibrium point sets S^3_ε at infinity. Now we are in a position to discuss the behaviour of the MIC–Kepler problem at infinity by projecting the trajectories of the conformal Kepler problem at infinity. After finding a stable and an unstable equilibrium point set at infinity for the MIC–Kepler problem, we will show that the trajectories of positive energy determine a map from the unstable equilibrium point set to the stable equilibrium point set.

From (3.5) and (4.2), points y in S^8 but outside the equator S^7 project to

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{1}{y_9^2} \begin{pmatrix} 2(y_1y_3 + y_2y_4) \\ 2(-y_1y_4 + y_2y_3) \\ y_1^2 + y_2^2 - y_3^2 - y_4^2 \end{pmatrix} \tag{5.1}$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{1}{2\sum_{j=1}^4 y_j^2} \begin{pmatrix} y_1y_7 + y_2y_8 + y_3y_5 + y_4y_6 \\ -y_1y_8 + y_2y_7 + y_3y_6 - y_4y_5 \\ y_1y_5 + y_2y_6 - y_3y_7 - y_4y_8 \end{pmatrix}. \tag{5.2}$$

Let $y(t)$ be a trajectory which has the limit point $\overline{y} \in S^3_\pm$ as $t \to \pm\infty$. Then it follows from (4.8), (5.1) and (5.2) that the projected trajectory $(\boldsymbol{q}(t), \boldsymbol{p}(t))$ satisfies

$$\frac{1}{r(t)} \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} \to (8h+1) \begin{pmatrix} 2(\overline{y}_1\overline{y}_3 + \overline{y}_2\overline{y}_4) \\ 2(-\overline{y}_1\overline{y}_4 + \overline{y}_2\overline{y}_3) \\ \overline{y}_1^2 + \overline{y}_2^2 - \overline{y}_3^2 - \overline{y}_4^2 \end{pmatrix} \tag{5.3}$$

$$\begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} \to \varepsilon \frac{\sqrt{8h}}{2}(8h+1) \begin{pmatrix} 2(\overline{y}_1\overline{y}_3 + \overline{y}_2\overline{y}_4) \\ 2(-\overline{y}_1\overline{y}_4 + \overline{y}_2\overline{y}_3) \\ \overline{y}_1^2 + \overline{y}_2^2 - \overline{y}_3^2 - \overline{y}_4^2 \end{pmatrix} \tag{5.4}$$

where use has been made of $r = \sum_{j=1}^{4} y_j^2/y_9^2$. Equations (5.3) and (5.4) imply that the scaled trajectory $(\boldsymbol{q}(t)/r(t), \boldsymbol{p}(t)) \in \boldsymbol{R}^6$ goes to the equilibrium set, as $t \to \pm\infty$,

$$S_\varepsilon^2 = \left\{ x \in \boldsymbol{R}^6 \middle| \sum_{k=1}^{3} x_k^2 = 1,\ x_{3+k} = \varepsilon \frac{\sqrt{8h}}{2} x_k, k = 1, 2, 3 \right\} \qquad \varepsilon = \pm \tag{5.5}$$

which are clearly diffeomorphic with S^2. The sets S_ε^2 are looked upon as the reduced space from S_ε^3 through the $SO(2)$ reduction as follows. The $SO(2)$ action (3.2) is naturally extended to that on S^8:

$$\begin{aligned} &(y_1 + \mathrm{i}y_2, y_3 + \mathrm{i}y_4) \mapsto \mathrm{e}^{\mathrm{i}t}(y_1 + \mathrm{i}y_2, y_3 + \mathrm{i}y_4) \\ &(y_5 + \mathrm{i}y_6, y_7 + \mathrm{i}y_8) \mapsto \mathrm{e}^{\mathrm{i}t}(y_5 + \mathrm{i}y_6, y_7 + \mathrm{i}y_8) \\ &y_9 \mapsto y_9. \end{aligned} \tag{5.6}$$

Since S_ε^3 with $\varepsilon = \pm$ are invariant under this action, we can take the factor space $S_\varepsilon^3/SO(2)$, which can be realized, in view of (5.1) and (5.2), as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (8h+1) \begin{pmatrix} 2(y_1y_3 + y_2y_4) \\ 2(-y_1y_4 + y_2y_3) \\ y_1^2 + y_2^2 - y_3^2 - y_4^2 \end{pmatrix} \tag{5.7}$$

$$\begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \frac{8h+1}{2} \begin{pmatrix} y_1y_7 + y_2y_8 + y_3y_5 + y_4y_6 \\ -y_1y_8 + y_2y_7 + y_3y_6 - y_4y_5 \\ y_1y_5 + y_2y_6 - y_3y_7 - y_4y_8 \end{pmatrix}. \tag{5.8}$$

Since $y \in S_\varepsilon^3$, equations (5.7) and (5.8) result in (5.5).

We now wish to discuss the relation between S_+^2 and S_-^2. These sets are viewed as equilibrium sets at infinity, which are linked together through trajectories $(\boldsymbol{q}(t), \boldsymbol{p}(t))$. Since trajectories are determined by the constants of motion, $\boldsymbol{J}$ and $\boldsymbol{A}$, we may assume that the constants of motion link S_+^2 and S_-^2. In what follows, we are working in the configuration space $\dot{\boldsymbol{R}}^3$. Let $\boldsymbol{q}(t)/r \to \boldsymbol{a}_\varepsilon$ as $t \to \pm\infty$. Then equations (5.3) and (5.4) show that $\boldsymbol{p}(t) \to \varepsilon\sqrt{2h}\boldsymbol{a}_\varepsilon$. Thus we observe that the trajectory leaves from infinity in the direction of $\boldsymbol{a}_-$ with the momentum $-\sqrt{2h}\boldsymbol{a}_-$ and finally tends to infinity in the direction of $\boldsymbol{a}_+$ with the momentum $\sqrt{2h}\boldsymbol{a}_+$. This fact allows the following physical interpretation: since the orbit $\boldsymbol{q}(t)$ is a hyperbola in the case of positive energy, the unit vector $\boldsymbol{q}(t)/r$ will have a definite direction as $t \to \pm\infty$, which is denoted by $\boldsymbol{a}_\varepsilon$ with $\varepsilon = \pm\infty$. Moreover, since $\boldsymbol{q}(t)/r$ has a definite direction, the momentum $\boldsymbol{p}(t)$ will have the same direction as $\boldsymbol{a}_+$ if $t \to \infty$ and the direction opposite to $\boldsymbol{a}_-$ if $t \to -\infty$, so that one has $\boldsymbol{p}(t) \to \varepsilon c\boldsymbol{a}_\varepsilon$ as $t \to \pm\infty$, where c is a positive constant. The c can be calculated by the use of the energy conservation which holds at infinity as well, $h = \frac{1}{2}|c\boldsymbol{a}_\varepsilon|^2$, with the result that $c = \sqrt{2h}$.

Since the constants of motion are conserved along the trajectory, we may take the limit of $\boldsymbol{A}$ as $t \to \pm\infty$. Thus we obtain, from (3.11),

$$\boldsymbol{A} = \boldsymbol{J} \times (\varepsilon\sqrt{2h}\boldsymbol{a}_\varepsilon) + \kappa\boldsymbol{a}_\varepsilon. \tag{5.9}$$

We denote by R the vector space isomorphism of $\boldsymbol{R}^3$ with $so(3)$, the Lie algebra of $SO(3)$,

$$R(\boldsymbol{u}) = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \qquad \boldsymbol{u} \in \boldsymbol{R}^3. \tag{5.10}$$

Then equation (5.9) yields

$$\boldsymbol{A} = (\kappa I + \sqrt{2h}R(\boldsymbol{J}))\boldsymbol{a}_+ = (\kappa I - \sqrt{2h}R(\boldsymbol{J}))\boldsymbol{a}_-. \tag{5.11}$$

This results in

$$a_+ = \left(I + \frac{\sqrt{2h}}{\kappa}R(J)\right)^{-1}\left(I - \frac{\sqrt{2h}}{\kappa}R(J)\right)a_-. \tag{5.12}$$

We observe further that for $h > 0$ the matrix on the right-hand side (5.12) gives rise to a Cayley transform $so(3) \to SO(3)$ (for a Cayley transform, see [8], for example):

$$\frac{\sqrt{2h}}{\kappa}R(J) \longmapsto S(J) := \left(I + \frac{\sqrt{2h}}{\kappa}R(J)\right)^{-1}\left(I - \frac{\sqrt{2h}}{\kappa}R(J)\right). \tag{5.13}$$

Summing up the above discussion, we have the following theorem.

Theorem 4. *For a fixed positive energy, the infinity of trajectories of the MIC–Kepler problem can be compactified to be the disjoint union of a stable and an unstable equilibrium point set, denoted by S^2_+ and S^2_-, respectively, both of which are diffeomorphic with S^2. Hyperbolic orbits determine a map of S^2_- to S^2_+, which is expressed as the rotation matrix $S(J)$ given in (5.13). Moreover, the map (5.13) provides a Cayley transform $so(3) \to SO(3)$, where $so(3)$ is viewed as a space of the angular momentum.*

The rotation matrix $S(J)$ may be called a scattering matrix. To describe $S(J)$ in the explicit form, we take a coordinate system in R^3 in such a way that $J = (0, 0, |J|)$. Then the matrix $S(J)$ is put in the form

$$S(J) = \frac{1}{1 + \frac{2h}{\kappa^2}|J|^2}\begin{pmatrix} 1 - \frac{2h}{\kappa^2}|J|^2 & \frac{2\sqrt{2h}}{\kappa}|J| & 0 \\ -\frac{2\sqrt{2h}}{\kappa}|J| & 1 - \frac{2h}{\kappa^2}|J|^2 & 0 \\ 0 & 0 & 1 + \frac{2h}{\kappa^2}|J|^2 \end{pmatrix}. \tag{5.14}$$

Here, we set

$$\cos 2\chi = \frac{1 - \frac{2h}{\kappa^2}|J|^2}{1 + \frac{2h}{\kappa^2}|J|^2} \qquad \sin 2\chi = \frac{\frac{2\sqrt{2h}}{\kappa}|J|}{1 + \frac{2h}{\kappa^2}|J|^2}. \tag{5.15}$$

Then equation (5.14) implies that $S(J)$ is a matrix describing a rotation about $J = |J|e_3$ by the angle 2χ. We note here that though $S(J)$ is a rotation matrix satisfying $a_+ = S(J)a_-$, the angle 2χ is not the angle between a_- and a_+ in the plane on which the orbit lies. Let us introduce here the projection operator P by

$$Px = x - \left(x \cdot \frac{J}{|J|}\right)\frac{J}{|J|} \qquad x \in R^3. \tag{5.16}$$

Since $S(J)J = J$, we can show that

$$S(J)Pa_- = Pa_+ \tag{5.17}$$

which means that the angle 2χ is the angle between Pa_- and Pa_+ in the plane perpendicular to J. Introducing here the angle $\theta_s := \pi - 2\chi$, we obtain the relation

$$\cot\frac{\theta_s}{2} = \tan\chi = \frac{\sqrt{2h}}{\kappa}|J|. \tag{5.18}$$

For $\mu = 0$, this formula reduces to a well known formula for the scattering angle of the Kepler problem [9].

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